A NOTE ON FUNCTIONS OF LINES

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A function of a line

$$F\overset{b}{\underset{a}{[y(x)]}} \tag{1}$$

may be regarded as a generalization of a function $F\ (y_1, y_2, \ldots, y_n)$ of a finite number of variables $y_i\ (i = 1, 2, \ldots, n)$. Instead of having a well defined value when a point $(y_1, y_2, \ldots, y_n)$ is given, the value of the function (1) is determined only when an infinitude of y-values belonging to an arc of the form

$$y = \varphi(x) \qquad (a \leqq x \leqq b) \tag{2}$$

is prescribed. The index i ranging over the integers $1, 2, \ldots, n$ in the function of a finite set of variables, is replaced in (1) by the index x ranging over the interval $a \leqq x \leqq b$. Examples of functions of this sort are the length of the arc (2), the time required by a heavy particle to fall from one end of the arc to the other, the area of the surface generated by revolving the arc about the x-axis, and many others.

For such functions Volterra has defined continuity and a derivative function.[1] The function (1) is said to be continuous at the arc (2) if for any given ϵ there always exists a δ such that

$$|\Delta F| = |F[\varphi(x) + \psi(x)] - F[\varphi(x)]| < \epsilon,$$

whenever $\psi(x)$ satisfies the conditions

$$|\psi(x)| < \delta \qquad (a \leqq x \leqq b).$$

Let $\psi\ (x)$ be further restricted not to change sign and to vanish identically except on an interval of length less than h containing a fixed value $x = \xi$. Then the derivative of F at the value ξ is defined by the equation

$$F'[\varphi(x), \xi] = \lim_{\substack{\delta = 0 \\ h = 0}} \frac{\Delta F}{\sigma},$$

where

$$\sigma = \int_a^b \psi(x) dx.$$

Further results of interest can be deduced[2] provided that the function F, the arc (2), and the value ξ, have associated with them a constant M such that

$$\left|\frac{\Delta F}{\delta h}\right| < M, \tag{3}$$

however $\delta > 0$, $h > 0$, and $\psi(x)$ are chosen, provided only that $\psi(x)$ is related to δ and h in the manner described above.

It is important that these considerations should apply to the integrals of the calculus of variations in terms of which the line functions cited above by way of illustration, with many others, are expressible. Such integrals in general are not continuous, do not possess derivatives, and do not satisfy the condition (3), in the forms specified by Volterra. It is the purpose of this note to prove this statement, and to call attention to the modifications of Volterra's definitions which apply also to the line functions of the calculus of variations.

Consider the simplest type of integrals of the calculus of variations.

$$F[y(x)] = \int_a^b f(x, y(x), y'(x))dx. \tag{4}$$

The length integral is a special case which is not continuous according to Volterra's definition. For in the figure $ac + cb$ is the length of each serrated line joining a with b and consisting of the slanting sides of the triangles with bases on ab and equal altitudes. There is one of these serrated paths in any neighborhood of the straight line ab. Hence the length integral is not continuous according to the definition given above.

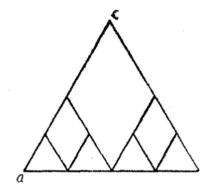

FIG. 1.

A function will be said to be of class $C^{(n)}$ if it is continuous and has continuous derivatives up to and including those of the n-th order. Consider then a curve (2) of class C'', and let the function f in the integral (4) also be of class C'' in a neighborhood R of the values (x,y,y') belonging to (2). Then the increment ΔF for the integral (4) is expressible in the form

$$\begin{aligned}\Delta F &= \int_a^b \{f(x, \varphi+\psi, \varphi'+\psi') - f(x, \varphi, \varphi')\}dx \\ &= \int_a^b \{A\psi + B\psi'\}dx,\end{aligned}$$

where[3]

$$A = \int_0^1 f_y(x, \varphi + \theta\psi, \varphi' + \theta\psi')d\theta,$$

$$B = \int_0^1 f_{y'}(x, \varphi + \theta\psi, \varphi' + \theta\psi')d\theta,$$

provided that ψ is continuous, and of class C'' except possibly at a finite number of values of x in the interval $a \leqq x \leqq b$, and provided also that the values $(x, \varphi + \psi, \varphi' + \psi')$ for $a \leqq x \leqq b$ are all in R. After the usual integration by parts of the calculus of variations, and an application of the mean value theorem for a definite integral, this becomes

$$\Delta F = \int_a^b \left(A - \frac{dB}{dx}\right) \psi\, dx = \left[A - \frac{dB}{dx}\right]_{x=x'} \int_a^b \psi(x)\, dx, \qquad (5)$$

where x' is a suitably selected value in the interval of length h or less including $x = \xi$, and on which ψ is not identically zero. Let ψ have the value

$$\psi = \sqrt{r^2 - (x - \xi)^2} - (r - \delta),$$

corresponding to the circular arc in Figure 2, on the interval

$$\xi - \frac{h}{2} \leqq x \leqq \xi + \frac{h}{2}$$

where it is not identically zero.

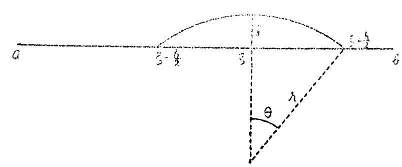

FIG. 2.

Then

$$\sigma = \int_a^b \psi\, dx = r^2 (\theta - \sin\theta \cos\theta), \quad \delta h = 2 r^2 \sin\theta\, (1 - \cos\theta).$$

If δ is allowed to approach zero, while r remains constant, it follows that h and θ both approach zero, and

$$\lim_{\substack{\delta=0 \\ h=0}} \frac{\Delta F}{\sigma} = \left[f_y - \frac{d}{dx} f_{y'} + \tfrac{1}{2} f_{y'y'} \frac{1}{r}\right]_{x=\xi}, \qquad (6)$$

$$\lim_{\substack{\delta=0 \\ h=0}} \frac{\Delta F}{\delta h} = \frac{4}{3}\left[f_y - \frac{d}{dx} f_{y'} + \tfrac{1}{2} f_{y'y'} \frac{1}{r}\right]_{x=\xi}, \qquad (7)$$

where the arguments in f and its derivatives are x, φ, φ'. It is clear from these expressions that at a value ξ defining a point on the curve (2) where the derivative $f_{y'y'}$ is different from zero, the limits (6) and (7) may take any arbitrarily assigned values, one value only being excepted in each case, provided that r is properly chosen. The function (4) has therefore no derivative in the sense of Volterra at the value $x = \xi$, and does not satisfy the condition (3).

For any more general variation ψ satisfying the restrictions specified in the first part of the last paragraph, it is clear from (5) that

$$\frac{\Delta F}{\sigma} = \left[\int_0^1 \left(f_y - \frac{d}{dx} f_{y'}\right) d\theta - \psi' \int^1 f_{y'y}\, \theta\, d\theta - \psi'' \int_0^1 f_{y'y'}\, \theta\, d\theta\right]_{x=x'}, \quad (8)$$

where the arguments of f and its derivatives are x, $\varphi + \theta\psi$, $\varphi' + \theta\psi'$. Hence if the conditions

$$|\psi(x)| < \delta, \quad |\psi'(x)| < \delta, \quad |\psi''(x)| < \delta \qquad (a \leqq x \leqq b), \quad (9)$$

as well as those described above are satisfied, the derivative limit will exist and have the value

$$\lim_{\substack{\delta=0\\ h=0}} \frac{\Delta F}{\sigma} = \left[f_y - \frac{d}{dx} f_{y'}\right]_{x=\xi},$$

the arguments of the derivatives of f being x, φ, φ'. Let N be the maximum of the absolute values of $f_y - df_{y'}/dx$, $f_{y'y}$, $f_{y'y'}$, in the neighborhood R. Then from (8)

$$\left|\frac{\Delta F}{\delta h}\right| = \left|\frac{\Delta F}{\sigma}\right| \left|\frac{\sigma}{\delta h}\right| \leqq \left|\frac{\Delta F}{\sigma}\right| \leqq N + N\frac{\delta}{2} + N\frac{\delta}{2},$$

and it is evident that the quotient $\Delta F/\delta h$ is bounded for all values of $\delta > 0$, $h > 0$, and ψ such that the inequalities (9) hold and the values $(x, \varphi + \psi, \varphi' + \psi')$ for $a \leqq x \leqq b$ are in R.

By similar arguments it will be clear what properties are possessed by an integral of the form

$$F[y(x)] = \int_a^b f(x, y, y', \ldots, y^{(n)})\, dx.$$

Let the curve (2) be defined by a function φ of class $C^{(2n)}$. In a neighborhood R of the values $(x, y, y', \ldots, y^{(n)})$ belonging to the curve, the function f is supposed to be of class $C^{(n+1)}$. Then[4] the function F has continuity of order n. In other words, for a given ϵ there always exists a δ such that

$$|\Delta F| = |F[\varphi(x) + \psi(x)] - F[\varphi(x)]| < \epsilon$$

whenever ψ is of class $C^{(n)}$, or continuous and of class $C^{(n)}$ except possibly at a finite number of x-values, and

$$|\psi(x)| < \delta, \; |\psi'(x)| < \delta, \; \ldots\ldots, \; |\psi^{(n)}(x)| < \delta \qquad (a \leqq x \leqq b). \qquad (10)$$

Further F has a derivative at any value $x = \xi$ which is approached with order $2n$; that is, if ψ does not change sign and vanishes except on an interval of length less than h including $x = \xi$, and if furthermore

$$|\psi(x)| < \delta, \; |\psi'(x)| < \delta, \; \ldots\ldots, \; |\psi^{(2n)}(x)| < \delta, \qquad (11)$$

then the limit

$$F'[\varphi(x), \xi] = \lim_{\substack{\delta = 0 \\ h = 0}} \frac{\Delta F}{\sigma}$$

exists. Further the absolute value of the quotient $\Delta F/\delta h$ will be bounded for all choices of $\delta > 0$, $h > 0$, $\psi(x)$ satisfying the relations (11) and such that the values $(x, y, y', \ldots, y^{(n)})$ on the arc $y = \varphi + \psi$, $a \leqq x \leqq b$, are all in the neighborhood R.

[1] Volterra, *Leçons sur les equations intégrales*, ch. 1, art. 5: or his *Leçons sur les fonctions des lignes*, ch. 1, art. 2.

[2] Volterra, arts. VII and 2, 3, respectively, of the chapters referred to above.

[3] See Jordan, *Cours d'Analyse*, vol. 1, p. 247.

[4] See Fischer, A generalization of Volterra's derivative of a function of a curve, *Amer. J. Math.*, **35**, 385 (1913).

A CLASSIFICATION OF QUADRATIC VECTOR FUNCTIONS

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There is probably no chapter of mathematics more worthy of attention, or more neglected at present, than the theory of vector functions. In the case of the linear vector function, it is true, a good deal has been found out in one way or another, and this by some of the very greatest of mathematicians. First investigated in detail by Hamilton[1] and again appearing as Cayley's matrix of the third order,[2] the linear vector function is essentially the same as the Grassmann open product[3] and the Gibbs dyadic.[4] In Germany the nonion or three-square matrix bears the name Tensor,[5] a word used by others in a different sense. On the other hand, we may make a clean sweep of all these